Tests of basic quantum mechanics in oscillation experiments

1999 J. Phys. A: Math. Gen. 32 4105

(http://iopscience.iop.org/0305-4470/32/22/311)

J. Phys. A: Math. Gen. **32** (1999) 4105–4110. Printed in the UK PII: S0305-4470(99)99510-0

Tests of basic quantum mechanics in oscillation experiments

G Domokos and S Kovesi-Domokos

The Henry A Rowland Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, MD 21218, USA

E-mail: SKD@HAAR.PHA.JHU.EDU

Received 20 November 1998, in final form 12 March 1999

Abstract. According to standard quantum theory, the time evolution operator of a quantum system is independent of the state of the system. One can, however, consider systems in which this is not the case: the evolution operator may depend on the density operator itself. The presence of such modifications of quantum theory can be tested in long-baseline oscillation experiments.

While quantum theory works remarkably well and there is no evidence to date that its validity may be limited, there have been attempts at modifying its structure both in order to resolve some conceptual problems or in order to establish its limits of validity. In particular, modifications of quantum theory have been considered in order to resolve the problem of transition to classical theory (the problem of 'decoherence'). Briefly, it has to be ensured that if two states are macroscopically different, then there is no definite phase relation between them and thus, their superposition is not a physically admissible state. (Sometimes this is dubbed as the 'problem of Schrödinger's cat', see for instance D'Espagnat [1].) It is generally thought that decoherence is a consequence of some kind of interaction with the environment in some general sense. For instance, it was conjectured that gravitational interactions cause decoherence once a body is sufficiently massive [2]. However, it is also conceivable that decoherence is an intrinsic property of quantum theory itself: after a sufficiently long time, phase relations are lost even between microscopic states and, therefore, macroscopic ones as well (*spontaneous decoherence*). Clearly, this requires a modification of the formalism of quantum theory as we know it. Such modifications have been repeatedly advocated by a variety of authors; a reasonably complete and up-to-date account of the problem of decoherence is given in the book by Omnès [3].

In this paper we consider one class of such possible modifications and suggest that one can place limits on departures from the standard version of quantum theory in experiments currently planned or being constructed.

The standard form of the time evolution operator of a quantum system described by its density operator, ρ is given by the expression†

$$\mathrm{i}\frac{\partial \rho}{\partial t} = [H, \rho] \tag{1}$$

where H is the Hamiltonian of the system. Equation (1) is linear in ρ. This fact has remarkable consequences; perhaps the most important one is that if at a particular time a system was in a

† Natural units are used: $\hbar = c = 1$

0305-4470/99/224105+06$19.50

pure state (i.e. $\rho^2(t_0) = \rho(t_0)$), then it will remain in a pure state at all times. (Conversely, a statistical mixture will not evolve spontaneously into a pure state either.)

It is conceivable, however, that due to the presence of some hitherto undetected small nonlinearity in equation (1), the property we have just described is only an approximate one: if left alone for a sufficiently long time, the system spontaneously evolves into a mixture or a mixture contracts to some pure state.

In order to explore this possibility, we add a nonlinear term to the right-hand side of the evolution equation, (1) and from now on investigate the evolution of a density operator obeying the equation:

$$\mathrm{i}\frac{\partial\rho}{\partial t} = [H, \rho] - \mathrm{i}\frac{\epsilon}{T}\left(f(\rho) - \frac{1}{N}\,\mathrm{Tr}\, f(\rho)\right). \tag{2}$$

Here T is some characteristic time scale. The constant ϵ is chosen to be ± 1: we shall presently see that the choice of the sign of the nonlinear term has a profound effect on the behaviour of the system. For that reason, it is useful to exhibit a sign factor explicitly. The function $f(\rho)$ governs the deviation of the evolution from standard quantum mechanics. Not having any firm guiding principle, we shall experiment with some simple functional forms. One hopes that by considering some simple examples, a pattern can be discovered in the behaviour of the solutions.

Here and for the rest of this paper we restrict ourselves to state spaces of dimension N. A generalization to infinite-dimensional state spaces appears to be feasible; however, we have not explored it in detail.

In writing down (2) we were guided by some physical prejudices. Three of those are worth noting.

- We want to maintain probability conservation, therefore the term added to the right-hand side of (1) is traceless. Consequently,

  $$\frac{\partial\,\mathrm{Tr}\,\rho}{\partial t} = 0$$

  as in standard quantum mechanics.
- We wrote down the evolution equation in such a manner that it contains a characteristic time scale governing the deviation from standard quantum mechanics. (Alternatively, we could have, for example, contemplated a deformation of the Heisenberg algebra. However, deformations of Lie algebras as discussed in the literature, contain dimensionless parameters. We consider T a constant of Nature; there are no dimensionless constants of Nature we know of†.)
- The term added to the evolution equation is local in time. One could have given, for example, some memory to the system by making the evolution equation depend on ρ taken at some past moment or upon an integral of ρ, etc, however, if eventually, one wants to construct manifestly Lorentz invariant theories, such terms are hard or impossible to incorporate.

By making the transformation,

$$\rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho_1(t)\,\mathrm{e}^{\mathrm{i}Ht} \tag{3}$$

the time dependence due to the Hamiltonian is eliminated; the quantity ρ_1 obeys the equation

$$\frac{\partial\rho_1}{\partial t} + \frac{\epsilon}{T}\left(f(\rho_1) - \frac{1}{N}\,\mathrm{Tr}\, f(\rho_1)\right) = 0. \tag{4}$$

† Often, the fine structure constant is considered a constant of Nature. It is not, however, because its magnitude depends on the momentum scale at which the measurement is performed.

We notice that 'total disorder', i.e.

$$\rho_1 = \frac{1}{N} \tag{5}$$

is a zero of $f - 1/N \operatorname{Tr} f$. Whether or not it is also an attractor, depends on the functional form of f. (One can linearize around (5) in order to determine this; however, the linearized version gives no information about the size of the basin of attraction.) Due to the vanishing of $f - 1/N \operatorname{Tr} f$, the density matrix in equation (5) is stationary.

In order to make further progress, we now consider some simple examples. The main technical simplification introduced is that we further restrict the dimensionality of the state space: we take $N = 2$; in this way, we can take advantage of the properties of the algebra of quaternions (equivalently, of the Pauli matrices).

In two dimensions, a density matrix is of the form,

$$\rho = \tfrac{1}{2}(1 + \boldsymbol{s} \cdot \boldsymbol{\sigma}) \tag{6}$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrices and $\boldsymbol{s}^2 \leqslant 1$. Clearly, since ρ and ρ_1 are unitarily equivalent, ρ_1 can be written in the same form as (6), namely

$$\rho_1 = \tfrac{1}{2}(1 + \boldsymbol{s}_1 \cdot \boldsymbol{\sigma}).$$

Consider now,

Example 1.

$$f(\rho_1) = \rho_1^2. \tag{7}$$

Equation (4) can be solved immediately. We have

$$\boldsymbol{s}_1(t) = \boldsymbol{s}_1(0) \exp\left(\frac{-\epsilon t}{2T}\right). \tag{8}$$

Clearly, only $\epsilon > 0$ makes sense from the physical point of view, since $\boldsymbol{s}_1^2 \leqslant 1$.

Example 2.

$$f(\rho_1) = \rho_1^3. \tag{9}$$

Just as in the case of equation (7), the evolution equation (4) can be solved in a closed form. The equation reads

$$\frac{\partial \boldsymbol{s}_1}{\partial t} + \frac{\epsilon}{8T}\boldsymbol{s}_1\left(3 + \boldsymbol{s}_1^2\right) = 0. \tag{10}$$

Clearly, the direction of $\boldsymbol{s}_1$ is constant and by taking the scalar product of equation (10) with $\boldsymbol{s}_1$, one obtains an equation for the magnitude of the polarization†, namely

$$\frac{\partial \boldsymbol{s}_1^2}{\partial t} = \frac{-\epsilon}{4T}\boldsymbol{s}_1^2\left(3 + \boldsymbol{s}_1^2\right). \tag{11}$$

The solution of equation (11) is

$$\boldsymbol{s}_1^2(t) = \boldsymbol{s}_1^2(0)\, \mathrm{e}^{-3\epsilon t/4T}\left[3 + \boldsymbol{s}_1^2(0)\left(1 - \mathrm{e}^{-3\epsilon t/4T}\right)\right]^{-1}. \tag{12}$$

Clearly, only $\epsilon > 0$ is physically acceptable. Both examples considered so far are such that for physically acceptable values of the parameter ϵ, total disorder (equation (5)) is an attractor

† 'Polarization' is used in a generalized sense. In general, it is just a measure of the deviation from total disorder.

and after times of the order of T the quantum system will find itself in the neighbourhood of the attractor.

This need not be the case, however. One can construct examples with other attractors. It would be tempting to choose for f something like $\rho(1-\rho)$ in analogy with the logistics equation; presumably, such a term would drive the quantum system towards a pure state. However, that expression has no traceless part in two dimensions. One does not want any dimensionality to be singled out, thus some more complicated functional form has to be tried.

Example 3. Choose

$$f(\rho) = \rho^3 - \rho^2. \tag{13}$$

Clearly, $f(\rho) = 0$ for a pure state. Equation (4) can be solved in a closed form for this case too. As before, it is sufficient to give the time evolution of the magnitude of s_1. One has

$$s_1^2(t) = \frac{s_1^2(0)}{s_1^2(0) + \left(1 - s_1^2(0)\right) \mathrm{e}^{-\epsilon t/4T}}. \tag{14}$$

In this case, both positive and negative values of ϵ lead to physically acceptable results; however, the qualitative behaviour of the system depends crucially on the sign of ϵ.

(a) If $\epsilon > 0$, a pure state ($s^2 = 1$) is an attractor: if initially $s^2 < 1$, the system will move towards a pure state unless initially $s = 0$. Any point within the unit ball with the origin removed is within the basin of attraction of a pure state.
(b) The situation is reversed for $\epsilon < 0$: any point in the interior of the unit ball ($s^2 < 1$) is in the basis of attraction of total disorder (equation (5)) and the surface is an unstable fixed point.

This distinction is relevant from the point of view of experimental tests.

The possibility of testing for the presence of terms proportional to $1/T$ in the evolution equation of a quantum system arises from the fact that long-baseline neutrino oscillation experiments (MINOS, ICAROS, K2K, etc)—presently in the planning stage, or under construction—take place on length scales of the order of $d \simeq 10^3$ km; hence, they should be sensitive to characteristic times, $d = T \simeq 10^{-3}$ s. This is considerably larger than the time scales involved in typical terrestrial experiments. (For comparison, a typical atomic transition is characterized by times of the order of 1 $\mathrm{eV}^{-1} \simeq 10^{-15}$ s; the time associated with the $K_S - K_L$ mass difference is about $(\Delta m)^{-1} \simeq 2 \times 10^{-10}$ s.)

In principle, the test is a very simple one. We noticed that, as a consequence of probability conservation, a *total disorder* characterized by equation (5) is always a fixed point of equation (4), and hence of the complete density operator. From the examples considered, it is also likely that total disorder is an attractor unless matters are specially arranged, as in example 3. Therefore, a likely test for the presence of nonlinear terms in the evolution equation consists of a search for spontaneous depolarization as a function of time.

In order to make matters more quantitative, let us consider neutrino oscillations with a nonlinear term in the evolution equation discussed in example 1 above. Neutrinos are particularly advantageous from the point of view of testing for spontaneous decoherence, since their interaction with the environment is generally negligibly small. Thus, one may be able to distinguish between spontaneous and environmental [2–4] decoherence.

Consider therefore a Hamiltonian of a two-flavour system, say $(\nu_\mu, \nu_\mathrm{e})$ [5], with a Hamiltonian in the diagonal basis given as

$$H = \frac{E_1 + E_2}{2} + \frac{E_1 - E_2}{2}\sigma_3 \tag{15}$$

and a mixing matrix,

$$U = \mathrm{e}^{\mathrm{i}\theta\sigma_2}. \tag{16}$$

At momenta much higher than the rest masses of the neutrinos, the energies are given by the expressions

$$E_i \approx p + \frac{m_i^2}{2p} \qquad (i = 1, 2). \tag{17}$$

The density matrix in the flavour basis at production is given by

$$\rho(0) = \tfrac{1}{2}(1 + s_3(0)\,\sigma_3). \tag{18}$$

The value of $s_3(0)$ in equation (18) equals ± 1, depending on whether ν_μ or ν_e is produced. Using the preceding equations and equation (8), one readily obtains the density matrix at time t:

$$\rho(t) = \tfrac{1}{2} + \tfrac{1}{2}s_3(0)\,\mathrm{e}^{-t/T}\sigma_3\big(\cos^2\phi + \sin^2\phi\cos 4\theta\big) + \tfrac{1}{2}s_3(0)\,\mathrm{e}^{-t/T}\sin^2\phi(-\sigma_1 + \sigma_2), \tag{19}$$

where

$$\phi = \frac{\Delta m^2 t}{4p}.$$

(In the limit of $T \to \infty$, equation (19), of course, reproduces the standard result.) Similar expressions hold whenever total disorder is an attractor, as in the second example discussed above: in all such cases, the signature for a departure from standard quantum mechanics is a damping of the polarization.

Example 3 deserves special attention: the evolution equation has a fixed point at $s^2 = 1$. It is generally assumed that weak interactions produce neutrinos of a definite flavour, i.e. in a pure state. If the fixed point in example 3 is a stable one, one has virtually no chance of observing a deviation from standard quantum theory in a neutrino oscillation experiment, even though the evolution equation of the density matrix contains nonlinear terms. Even if the fixed point is unstable, one needs environmental perturbations in order to drive the neutrino away from a pure state: the relevant Lyapunov exponent may be too small to make the presence of a nonlinear term in the evolution equation observable. The lesson to be learnt from this example is that even though there may be deviations from standard quantum theory present in the evolution equation of the density operator, circumstances may conspire to effectively hide that deviation from experimental scrutiny.

A final remark is in order in this context. In less than a decade or so, there will be experimental data available from the long-baseline neutrino oscillation experiments mentioned above.

As suggested by the preceding discussion, results obtained from those experiments may provide a useful testing ground for the basic principles of quantum mechanics. However, in order to interpret the data from the point of view of testing quantum mechanics itself, care has to be taken in order to:

- Properly take environmental effects (flavour oscillations in a medium, beam damping in transit from the accelerator to the detector, etc) into account.
- In addition, one has to analyse plausible models of deviation from 'orthodox' quantum mechanics. We presented *some models* describing deviations from standard quantum theory. However, a general analysis of the robustness of such models is still missing. (We have, apparently, found one feature which appears to be a generic one: nonlinear quantum evolution equations appear to prefer 'chaos' as the final state of a quantum system.)

Further study is needed in order to establish whether or not the feature discovered here is a generic one, or merely a property of some specific models. One should note, in particular, that the classification of systems of nonlinear evolution equations is a difficult subject and it is still far from being completely understood.

To summarize, long-baseline neutrino oscillation experiments are likely to provide an environment for testing the validity of standard quantum theory, due to the unusually long distances involved in such experiments. From the examples considered here, it appears that, if nonlinear terms are present in the evolution equation for the density matrix, they are likely to lead to spontaneous decoherence. Nevertheless, some caution is needed: there may be situations in which the presence of deviations from standard quantum theory is hidden from observation in certain experiments.

Acknowledgments

This work was done during the authors' visit at the Dipartimento di Fisica, Universitá di Firenze. We thank Roberto Casalbuoni, Director of the Department, for the hospitality extended to us. We also thank Frigyes Károlyházy for several tutorial sessions on the problem of decoherence.

References

[1] D'Espagnat B 1976 *Conceptual Foundations of Quantum Mechanics* 2nd edn (Menlo Park, CA: Benjamin-Cummings) ch 3
[2] Károlyházy F 1966 *Nuovo Cimento* **42** 390
See also, Károlyházy F 1990 *Sixty Two Years of Uncertainty* ed A I Miller (New York: Plenum)
[3] Omnès R 1994 *The Interpretation of Quantum Mechanics* (Princeton, NJ: Princeton University Press) ch 7
[4] See, for example, Zurek W H 1982 *Phys. Rev.* D **26** 1862
See also Zurek W H 1991 *Phys. Today* **44** 36
[5] Kim C W and Pevsner A 1993 *Neutrinos in Physics and Astrophysics* (Geneva: Harwood) ch 7